

# EXACT SIMULATION OF BROWN-RESNICK RANDOM FIELDS AT A FINITE NUMBER OF LOCATIONS

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**ABSTRACT.** We propose an exact simulation method for Brown-Resnick random fields, building on new representations for these stationary max-stable fields. The main idea is to apply suitable changes of measure.

## 1. INTRODUCTION

Max-stable random fields are fundamental models for spatial extremes. These models have been coined by de Haan [8], and have recently found applications to extreme meteorological events such as rainfall modeling and extreme temperatures (Buishand et al. [2], de Haan and Zhou [9], Dombry et al. [5], Davis et al. [3], Huser and Davison [10]). There are three different kinds of normalized max-stable processes, with Gumbel, Fréchet, and Weibull marginals, respectively. In what follows, we restrict ourselves to max-stable processes with Gumbel marginals; corresponding results for Fréchet and Weibull marginals can be obtained by a monotone transformation of the Gumbel case.

This paper studies a particular class of max-stable random fields known as *Brown-Resnick random fields*. Simulation of these and related processes is complicated, and the literature exclusively focuses on approximate simulation techniques; see for example Schlather [20], Oesting et al. [15], Engelke et al. [7], Oesting and Schlather [16], Dombry et al. [5].

This paper is the first to devise an *exact* simulation method for Brown-Resnick random fields. The key ingredient is a new representation for Brown-Resnick random fields, which is of independent interest. In fact, we show that there is an uncountable *family* of representations. At the heart of our derivation of these representations lies a change of measure argument.

We now describe the results in this paper in more detail. For some index set  $T \subset \mathbb{R}^d$ , the process  $(Y(t))_{t \in T}$  of real-valued random variables is *max-stable* (with Gumbel marginals) if for a sequence of iid copies  $(Y^{(i)}(t))_{t \in T}$ ,  $i = 1, 2, \dots$ , of  $(Y(t))_{t \in T}$  the following relation holds

$$\left( \max_{i=1, \dots, n} Y^{(i)}(t) - \log n \right)_{t \in T} \stackrel{d}{=} (Y(t))_{t \in T}, \quad n \geq 1,$$

where this relation is interpreted in the sense of equality of the finite-dimensional distributions. Then, in particular, all one-dimensional marginals of the process  $(Y(t))_{t \in T}$  are Gumbel distributed, i.e.,  $Y(t)$  has distribution function  $\Lambda(x - c(t)) = \exp(-e^{-(x - c(t))})$ ,  $x \in \mathbb{R}$ , for some function  $c(t) \in \mathbb{R}$ ,  $t \in T$ . Throughout this paper, we work with  $T = \mathbb{R}^d$ .

In this paper, we consider a class of max-stable processes with representation

$$(1.1) \quad \eta(t) = \sup_{i \geq 1} (V_i + W_i(t) - \sigma^2(t)/2), \quad t \in \mathbb{R}^d,$$

where  $\sigma^2(t) = \text{Var}(W_1(t))$ ,  $t \in \mathbb{R}^d$ ,  $(W_i)$  is a sequence of iid centered Gaussian processes with stationary increments on  $\mathbb{R}^d$ , and  $(V_i)$  are the points of a Poisson process on  $\mathbb{R}$  with intensity measure  $e^{-x} dx$ . In the case of Brownian motions  $(W_i)$ , the process (1.1) was considered by Brown

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and Resnick [1] and shown to be stationary. It is common to refer to the more general model (1.1) as *Brown-Resnick random field* as well.

The representation (1.1) is not particularly suitable for exact sampling. Although  $(V_i)$  and  $(W_i)$  are easily simulated, it turns out that the naive simulation approach of replacing  $\sup_{i \geq 1}$  by  $\sup_{i \leq N}$  for some large  $N$ , may fail. For example, assume that  $W_i$  is standard Brownian motion on  $\mathbb{R}$ . Then, in view of the law of the iterated logarithm, each of the processes  $W_i(t) - \sigma^2(t)/2$  drifts to  $-\infty$  a.s. as  $t \rightarrow \infty$ . In turn, the process  $\sup_{i \leq N} (V_i + W_i(t) - \sigma^2(t)/2)$  drifts to  $-\infty$  as  $t \rightarrow \infty$  as well. In particular, the simulation of  $\eta$  requires an increasing number  $N$  if one aims at a sample path of the process on a larger interval. More importantly, it is unclear how  $N$  should be chosen.

Using our new representations, we obtain an exact sampling method for  $\eta$  at the points  $t_1, \dots, t_n \in \mathbb{R}^d$ , meaning that the output of the method has the same distribution as  $(\eta(t_1), \dots, \eta(t_n))$ . In our method, it is no longer problematic that the processes  $W_i(t) - \sigma^2(t)/2$  drift away to  $-\infty$  and a truncation point is automatically identified by our algorithm.

Several properties of Brown-Resnick processes readily follow from our representations, although they are not straightforward to see from (1.1). For instance, the process  $\eta$  is stationary in the sense that  $\eta$  has the same distribution as  $\eta(\cdot + c)$  for any choice of  $c \in \mathbb{R}^d$ . The process  $\eta$  also has standard Gumbel marginals. In deriving our representations from (1.1),  $\sigma^2$  drops out and we recover the known fact that the law of  $\eta$  only depends on the *variogram*

$$\gamma(t) = \frac{1}{2} \mathbb{E}(W(t) - W(0))^2, \quad t \in \mathbb{R}^d.$$

These properties were proved in Kabluchko et al. [12] with arguably more elaborate techniques.

Given an exact simulation method for the Brown-Resnick process with Gumbel marginals, we also have an exact simulation method for this process with Fréchet or Weibull marginals. For example, the processes  $e^\eta$  and  $-e^{-\eta}$  have Fréchet  $\Phi_1(x) = e^{-x^{-1}}$ ,  $x > 0$ , and Weibull  $\Psi_1(x) = e^{-|x|}$ ,  $x < 0$ , marginals, respectively.

**Notation.** We use the symbols  $W_1, W_2, \dots$  for iid centered Gaussian random fields with stationary increments, variance function  $\sigma^2$ , and variogram  $\gamma$ . We use the symbols  $Z_1, Z_2, \dots$  for iid Gaussian random fields with stationary increments, mean function  $-\gamma$ , variance function  $2\gamma$ , variogram  $\gamma$ , and vanishing at the origin. A generic copy of these fields is denoted by  $W$  and  $Z$ , respectively.

## 2. REPRESENTATIONS

In this section we provide new representations for the Brown-Resnick random field  $\eta$  given in (1.1). These representations arise from a change of measure. We make the same assumptions on the stationary Brown-Resnick process as in the previous section. All proofs for this section are in Section 5. We fix the functions  $\sigma^2$  and  $\gamma$  throughout this section.

The following theorem is the main result of this section.

**Theorem 2.1.** *Suppose we are given an arbitrary probability measure  $\mu$  on  $\mathbb{R}^d$ . Consider*

$$\zeta(t) = \sup_{i \geq 1} \left( V_i + Z_i(t - T_i) - \log \left( \int_{\mathbb{R}^d} \exp(Z_i(s - T_i)) \mu(ds) \right) \right), \quad t \in \mathbb{R}^d,$$

where  $((T_i, V_i))_{i \geq 1}$  are the points of a Poisson process on  $\mathbb{R}^d \times \mathbb{R}$  with intensity measure  $\mu(dt) \times e^{-v} dv$ . Then the random fields  $(\eta(t))_{t \in \mathbb{R}^d}$  and  $(\zeta(t))_{t \in \mathbb{R}^d}$  have the same distribution.

**Remark 2.2.** There is a continuum of random fields with the same distribution as  $\eta$ , one for each measure  $\mu$ .

**Remark 2.3.** Under the assumptions of Theorem 2.1, it is known that the field  $(\zeta'(t))_{t \in \mathbb{R}^d}$  with

$$\zeta'(t) = \sup_{i \geq 1} \left( V_i + Z_i(t - T_i) \right), \quad t \in \mathbb{R}^d$$

also has the same distribution as  $(\eta(t))_{t \in \mathbb{R}}$ . Although the fields  $\zeta$  and  $\zeta'$  differ due to the additional log-term, the theorem states they have the same distribution. This surprising fact becomes perhaps more plausible after noting that, for every  $i$ ,

$$\log \mathbb{E} \left( \int_{\mathbb{R}^d} \exp(Z_i(s - T_i)) \mu(ds) \right) = 0.$$

**Remark 2.4.** If  $\sigma^2/2 = \gamma$ , then  $(W_i(t - T_i))$  has the same distribution as  $(W_i(t) - W_i(T_i))$ . The term  $W_i(T_i)$  drops out of the expression for  $\zeta$ , so in that case the random field

$$\sup_{i \geq 1} \left( V_i + W_i(t) - \gamma(t - T_i) - \log \left( \int_{\mathbb{R}^d} \exp(W_i(s) - \gamma(s - T_i)) \mu(ds) \right) \right), \quad t \in \mathbb{R}^d,$$

also has the same distribution as  $(\eta(t))_{t \in \mathbb{R}^d}$ .

**Remark 2.5.** Oesting et al. [15, 17] provided various alternative point process representations of Brown-Resnick random fields. These representations are different from ours, although they appear similar in spirit. The paper [15] proposed to introduce random time shifts of the processes  $W_i$  and used this idea to derive approximate sampling methods for  $\eta$ . The paper [17] focused on a much wider class of max-stable processes than this paper.

Theorem 2.1 leads to the following three well known facts proved in Kabluchko et al. [12].

**Corollary 2.6.** *The field  $\eta$  is stationary.*

*Proof.* Let  $\mu$  be a Dirac point mass at some arbitrary  $t^* \in \mathbb{R}^d$ . Theorem 2.1 implies that the random field  $(\sup_{i \geq 1} (V_i + Z_i(t - t^*)))_{t \in \mathbb{R}^d}$  has the same distribution as  $(\eta(t))_{t \in \mathbb{R}^d}$ . In particular, the distribution does not depend on  $t^*$ .  $\square$

**Corollary 2.7.** *The one-dimensional marginals of  $(\eta(t))_{t \in \mathbb{R}^d}$  have the Gumbel distribution.*

*Proof.* If we let  $\mu$  be a point mass as in the proof of the preceding corollary, then we find that  $\eta(t)$  has the same distribution as  $\sup_{i \geq 1} V_i$  for every  $t \in \mathbb{R}^d$ .  $\square$

**Corollary 2.8.** *The distribution of  $(\eta(t))_{t \in \mathbb{R}^d}$  only depends on the variogram  $\gamma$ .*

*Proof.* Since the processes  $Z_i$  are completely determined by  $\gamma$ , the law of  $(\zeta(t))_{t \in \mathbb{R}^d}$  depends only on  $\gamma$ . Theorem 2.1 therefore immediately yields the claim.  $\square$

There are some interesting connections between Brown-Resnick random fields and familiar quantities in extreme value theory, which simply follow from the known finite-dimensional distribution functions of such fields. Details on this distribution function can be found in Section 5 (specifically Lemma 5.1); for now, we note that if  $\eta$  is stochastically continuous we have, for any  $N > 0$ ,

$$\mathbb{P} \left( \sup_{t \in [0, N]^d} \eta(t) \leq x \right) = \exp \left( -e^{-x} \mathbb{E} \exp \left( \sup_{t \in [0, N]^d} Z(t) \right) \right), \quad x \in \mathbb{R},$$

and therefore

$$\mathbb{P} \left( \sup_{t \in [0, N]^d} \eta(t) - d \log N \leq x \right) = \exp \left( -e^{-x} N^{-d} \mathbb{E} \exp \left( \sup_{t \in [0, N]^d} Z(t) \right) \right), \quad x \in \mathbb{R}.$$

Dieker and Yakir [4, Cor. 1], show that the set function

$$f(A) = \mathbb{E} \exp \left( \sup_{t \in A} Z(t) \right), \quad A \subset \mathbb{R}^d$$

is translation invariant:  $f(A) = f(t + A)$  for  $t \in \mathbb{R}^d$ ; this also follows from Corollary 2.6. (They only write out the one-dimensional fractional Brownian motion case, but the more general case follows from exactly the same arguments; it is based on Lemma 5.2 below.) Moreover,  $f$  is subadditive in the sense that  $f(A_1 \cup A_2) \leq f(A_1) + f(A_2)$  for disjoint subsets  $A_1, A_2 \subset \mathbb{R}^d$ . A basic fact about

such functions (e.g., Xanh [21]) is that  $f(A)$  grows like the Lebesgue measure of  $A$  for large sets  $A$ . In particular, this result implies that the limit

$$\lim_{N \rightarrow \infty} N^{-d} \mathbb{E} \exp \left( \sup_{t \in [0, N]^d} Z(t) \right)$$

exists. This quantity is known as *Pickands's constant*; we refer to the monograph by Piterbarg [18] for an extensive discussion of these quantities. The numerical determination of this constant and the simulation of the Brown-Resnick process  $\eta$  suffer from the same problems mentioned in the Introduction. Dieker and Yakir [4] proposed a Monte Carlo method for determining the Pickands constant.

The discrete analogs of Pickands's constant are connected to extremal indices of the Brown-Resnick processes. Assume  $d = 1$  and consider a Brown-Resnick process  $(\eta(t))_{t \in \mathbb{R}}$ . Its restriction to the integers yields a strictly stationary time series  $(\eta(i))_{i \in \mathbb{Z}}$ . For  $x \in \mathbb{R}$  we have

$$\mathbb{P} \left( \max_{i=1, \dots, n} \eta(i) - \log n \leq x \right) = \exp \left( -e^{-x} n^{-1} \mathbb{E} \left[ \max_{i=1, \dots, n} e^{Z(i)} \right] \right).$$

This leads to the limit relation

$$\lim_{n \rightarrow \infty} \mathbb{P} \left( \max_{i=1, \dots, n} \eta(i) - \log n \leq x \right) = \Lambda^\theta(x), \quad x \in \mathbb{R},$$

where the limit

$$\theta = \lim_{n \rightarrow \infty} n^{-1} \mathbb{E} \left[ \max_{i=1, \dots, n} e^{Z(i)} \right]$$

exists by subadditivity and translation invariance as in the continuous case. It is well known (see Leadbetter et al. [14], cf. Section 8.1 in Embrechts et al. [6]) that  $\theta$  is a number in  $[0, 1]$ . The quantity  $\theta$  is the *extremal index* of the stationary sequence  $(\eta(i))_{i \in \mathbb{Z}}$ . The reciprocal of this quantity is often interpreted as the expected value of the cluster size of high-level exceedances of the sequence  $(X_i)$ ; see for example [14]; cf. Section 8.1 in [6]. The constant  $\theta$  appears in Dieker and Yakir [4] as a special case of the constants  $\eta \mathcal{H}_\alpha^\eta$ ; see Proposition 3 in [4] for a characterization alternative to the extremal index. Although we do not have a proof that  $\theta$  is smaller than Pickands's constant in the continuous-time case, simulation evidence indicates that this fact is true.

### 3. A SIMULATION ALGORITHM

This section presents a simulation algorithm for Brown-Resnick random fields on a discrete set of points  $t_1, \dots, t_n \in \mathbb{R}^d$ . We may assume that  $\sigma^2/2 = \gamma$  in this section. Since Theorem 2.1 gives a different representation for each choice of  $\mu$ , it would be interesting to know which choice leads to the fastest algorithm. Here we simply let  $\mu$  be uniform on  $\{t_1, \dots, t_n\}$ .

Remark 2.4 shows that the vector  $(N(t_1), \dots, N(t_n))$  with, for  $j = 1, \dots, n$ ,

$$N(t_j) = \sup_{i \geq 1} \left( V_i + W_i(t_j) - \gamma(t_j - T_i) - \log \left( n^{-1} \sum_{\ell=1}^n \exp(W_i(t_\ell) - \gamma(t_\ell - T_i)) \right) \right)$$

has the same distribution as  $(\eta(t_1), \dots, \eta(t_n))$ , where  $((V_i, T_i))_{i \geq 1}$  belong to a Poisson process on  $\mathbb{R} \times \{t_1, \dots, t_n\}$  with intensity measure  $e^{-x} dx \times (n^{-1} \sum_{i=1}^n \delta_{t_i}(dy))$ . We slightly rewrite the above display as

$$N(t_j) = \sup_{i \geq 1} \left( V_i + \log n + W_i(t_j) - \gamma(t_j - T_i) - \log \left( \sum_{\ell=1}^n \exp(W_i(t_\ell) - \gamma(t_\ell - T_i)) \right) \right).$$

This is the representation we use for our simulation algorithm.

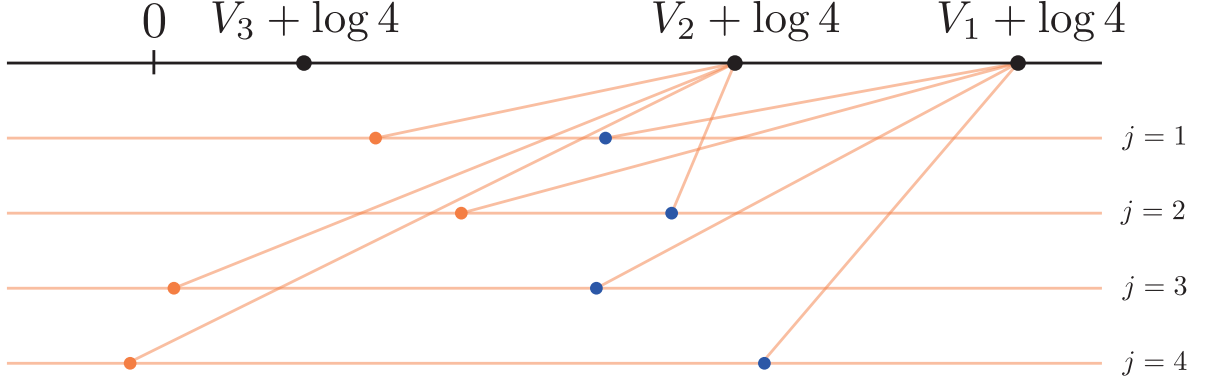


FIGURE 1. Illustration of our algorithm for  $n = 4$ . The points  $V_i + \log 4$  are generated in decreasing order. Each ‘level’ below the axis represents a value of  $j$ , and each  $(V_i + \log 4)$ -point is connected to its cluster points  $C_i(t_j)$ . The cluster points  $C_i(t_j)$  always lie to the left of  $V_i + \log 4$ .

A point  $V_i$  on  $\mathbb{R}$  gives rise to a ‘cluster’ of points  $\{C_i(t_j) : j = 1, \dots, n\}$  with

$$C_i(t_j) = (V_i + \log n) + W_i(t_j) - \gamma(t_j - T_i) - \log \left( \sum_{\ell=1}^n \exp(W_i(t_\ell) - \gamma(t_\ell - T_i)) \right).$$

These cluster points can be visualized by interpreting them as belonging to different levels depending on the value of  $j$ ; see Figure 1. The variable  $N(t_j) = \sup_{i \geq 1} C_i(t_j)$  is then the maximum of all cluster points on the  $j$ -th level. The crucial insight is that *only a finite number of points/cluster pairs  $(V_i, C_i)$  need to be generated, since  $C_i(t_j) \leq V_i + \log n$  and we seek  $\sup_{i \geq 1} C_i(t_j)$  for  $j = 1, \dots, n$ .* The algorithm generates points/cluster pairs  $(V + \log n, C)$  in decreasing order of  $(V + \log n)$ -value, until the next  $(V + \log n)$ -value is smaller than the current maximum over the cluster points on each level. For instance, in Figure 1, after  $V_3 + \log 4$  has been generated, none of the remaining cluster points can change the values of  $(N(t_1), \dots, N(t_4))$ , which have been given a different color.

To get a sense of how many points of  $V$  will be generated, let us consider the (degenerate) case where  $t_1 = \dots = t_n = t$ . We then have  $C_i(t_j) = V_i$  for  $j = 1, \dots, n$ , so the algorithm terminates after generating  $\inf\{M : V_M + \log n < V_1\}$  points of  $V$ . For large  $n$ , this implies that the number of points is of order  $n$ .

We remark that this algorithm is suitable for parallelization. Indeed, several points of the  $V$ -process can be generated simultaneously instead of one at the time, with corresponding clusters being computed on different processors. Specifically, with one master and  $K$  workers, the algorithm would consist of a number of steps, each of which computes the next  $K$  clusters in parallel. At each step of the algorithm, the master generates the next  $K$   $V$ -points in decreasing order. This is readily done since the points  $(e^{-V_i})$  constitute a standard Poisson process on  $\mathbb{R}_+$ . Each of the  $K$  clusters would then be computed on a worker node, after which the master checks whether the algorithm can be terminated or whether further steps are needed.

#### 4. NUMERICAL EXPERIMENTS

This section reports on several simulation experiments we have carried out in order to validate our algorithm and to test its performance in terms of speed. Throughout, we work with Brown-Resnick random fields with variogram  $\gamma(t) = |t|^\alpha/2$  for some  $\alpha \in (0, 2]$ . Appendix A has some implementation details.

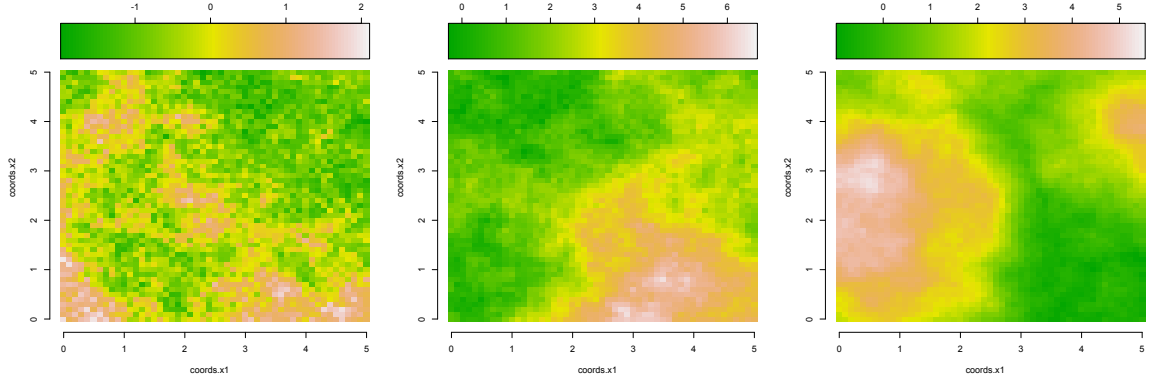


FIGURE 2. Sample of a Brown-Resnick random field on  $[0, 5]^2$  with variogram  $\gamma(t) = |t|^\alpha/2$  for  $\alpha = 1/2$ ,  $\alpha = 1$ ,  $\alpha = 3/2$  from left to right, respectively. The grid mesh is 0.1.

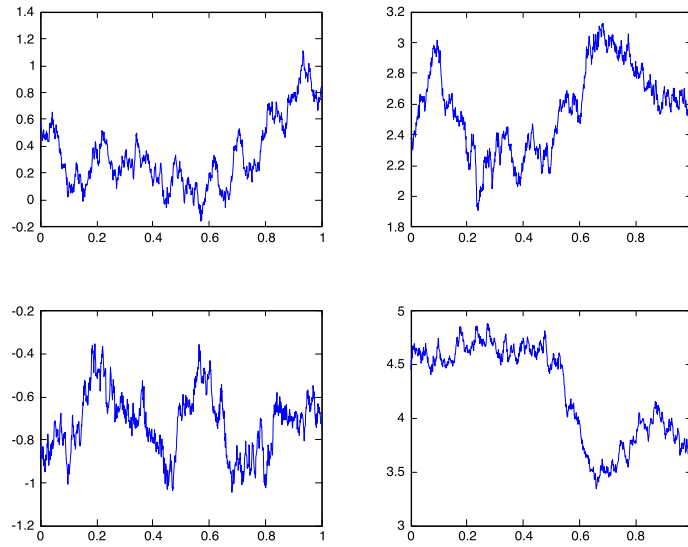


FIGURE 3. Representative samples of a Brown-Resnick process on  $[0, 1]$  with variogram  $\gamma(t) = |t|/2$ .

**Representative samples.** We have implemented the algorithm in R (see [19]) in order to leverage the existing toolkit to generate the Gaussian random fields that are needed in our algorithm. We use the R package `RandomFields` by Schlather et al., which is available through R's package manager. Three representative samples of Brown-Resnick random fields are given in Figure 2, with various levels of a smoothness parameter  $\alpha$ . We see that the paths become rougher as  $\alpha$  decreases, as it should be. The random field is the maximum of random 'mountains' (given by quadratic forms) if  $\alpha = 2$ , and our replication for  $\alpha = 3/2$  exhibits similar behavior in the sense that two mountains can be distinguished.

In the rest of this section, we carry out all experiments in the one-dimensional case  $d = 1$  for computational ease. Figure 3 depicts some representative one-dimensional samples for  $\alpha = 1$ . Note that it indeed appears that these are realizations of a stationary process even though our algorithm

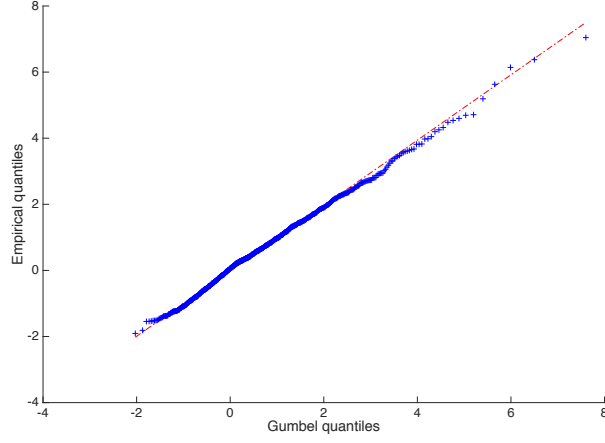


FIGURE 4. Q–Q plot illustrating that our samples of  $\eta(0) \vee \eta(s) - \log(2\Phi(\sqrt{s}/2))$  have a standard Gumbel distribution for  $s = 1 - 1/1024$ .

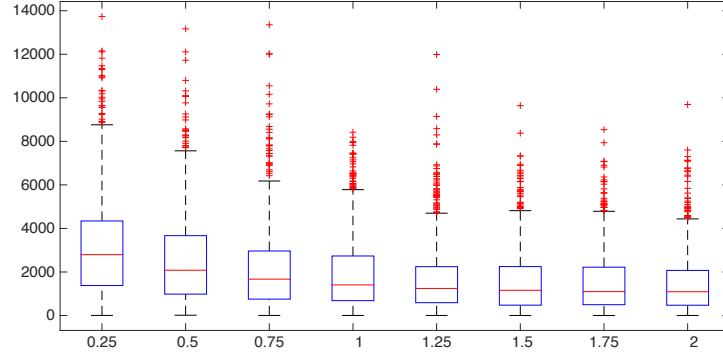


FIGURE 5. Box plot showing the number of clusters generated as a function of  $\alpha$ , with  $n = 1024$ . The edges of the box are the 25th and 75th percentiles.

does not require truncating the number of Gaussian random field samples if one aims at a sample path of the process on a larger interval.

**Dependence structure.** We next verify whether our simulation algorithm captures the dependence within the process correctly. To this end, we generated 1000 samples of  $\eta(0) \vee \eta(s)$  for  $\alpha = 1$  in the one-dimensional case. This random variable has a (nonstandard) Gumbel distribution.

We note that

$$\begin{aligned} & -\log \mathbb{P}(\eta(0) \vee \eta(s) \leq x) \\ &= e^{-x} \left[ \mathbb{P}(W(s) \leq s/2) + \mathbb{E}(e^{W(s)-s/2}; W(s) > s/2) \right] \\ &= e^{-(x - \log(2\Phi(\sqrt{s}/2)))}, \end{aligned}$$

where  $\Phi$  is the distribution function of the standard normal distribution. The resulting Q–Q plot for  $s = 1 - 1/1024$  is given in Figure 4.

**Number of clusters.** We next investigate numerically whether the dependency structure influences the number of points  $V_i$  that are generated by our algorithm for a single replication of the Brown-Resnick process. To do so, we generated 1000 replications of  $(\eta(0), \eta(1/1024), \dots, \eta(1 - 1/1024))$  for various values of  $\alpha$ . Figure 5 summarizes the results in a box plot.

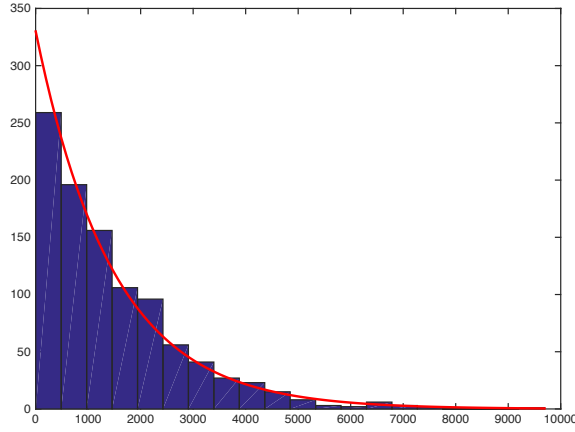


FIGURE 6. Histogram of the number of  $V_i$  points for  $\alpha = 2$ , compared with an exponential density.

The data provides evidence that rougher paths are harder to simulate, which suggests that the order  $n$  bound derived in Section 3 is in fact a lower bound on the number of  $V_i$  points that need to be generated. In the code used for Figure 5, we preprocess some of the computations required for sampling the  $W_i$ . This results in significant savings. We have not included this code in Appendix A for expository reasons.

We next compare the histogram of the number of  $V_i$  points for  $\alpha = 2$  with a fitted exponential density, see Figure 6. This figure provides evidence that this distribution has light tails. For other values of  $\alpha$ , the corresponding histograms also indicate light tails, although the distribution looks more like a gamma distribution.

**Speed.** The speed of our algorithm in practice heavily depends on how quickly the underlying Gaussian random fields can be generated. In our one-dimensional case, we generate the Gaussian processes with the recent Matlab implementation by Kroese and Botev [13]. Theoretically, the computational effort needed to generate a sample from the underlying Gaussian process is independent of  $\alpha$  in this implementation. Thus, the running time depends linearly on the number of points  $V_i$  that are generated by the algorithm, which is different for different samples.

In Matlab it is difficult to record CPU time (as opposed to elapsed time), and we have observed wide variation (up to 50%) in run time with exactly the same random input on a dedicated CPU. Thus, we keep the discussion at a high level. For the experiment reported in Figure 5 with  $n = 1024$ , each sample is generated in the order of seconds on a single core of a 2.7 GHz Intel Core i7 processor regardless the value of  $\alpha$ , with most runs less than a second and a few runs more than four seconds.

## 5. PROOFS

This section presents the proof of Theorem 2.1 We fix the functions  $\sigma^2$  and  $\gamma$  throughout this section. Contrary to the preceding two sections, we do not assume that  $\gamma = \sigma^2/2$  but we shall see that the function  $\sigma^2$  vanishes from our calculations.

We start with an auxiliary lemma; see de Haan [8] and Kabluchko et al. [12] for proofs.

**Lemma 5.1.** *Let  $(X_i)$  be iid copies of some random field  $X$  on  $\mathbb{R}^d$  and  $(V_i)$  the points of a Poisson process on  $\mathbb{R}$  with intensity measure  $e^{-x} dx$ . If we write*

$$\xi(t) = \sup_{i \geq 1} (V_i + X_i(t)), \quad t \in \mathbb{R}^d,$$



then we have for  $y_j \in \mathbb{R}, t_j \in \mathbb{R}^d, i = 1, \dots, n$ ,

$$\mathbb{P}(\xi(t_1) \leq y_1, \dots, \xi(t_n) \leq y_n) = \exp \left( - \mathbb{E} \exp \left( \max_{j=1, \dots, n} (X(t_j) - y_j) \right) \right).$$

The following change of measure lemma plays a key role in our argument, and shows why the variance function  $\sigma^2$  vanishes from the calculations. It is a field version of Lemma 1 in Dieker and Yakir [4], see also [11, Prop. 2] for the underlying change of measure result. We only sketch the key idea of the proof insofar as it highlights the differences with [4], since the lemma follows from the same arguments as given there.

**Lemma 5.2.** *Fix  $t \in \mathbb{R}^d$ . For a measurable functional  $F$  on  $(\mathbb{R}^d)^\mathbb{R}$  that is translation invariant, we have*

$$\mathbb{E} e^{W(t) - \sigma^2(t)/2} F(W - \sigma^2/2) = \mathbb{E} F(\theta_t Z),$$

where the shift  $\theta_t$  is defined through  $(\theta_t Z)(s) = Z(s - t)$ .

*Proof sketch.* Set  $\mathbb{Q}(A) = \mathbb{E}[e^{W(t) - \sigma^2(t)/2} 1_A]$ , and write  $\mathbb{E}^\mathbb{Q}$  for the expectation operator with respect to  $\mathbb{Q}$ . In this sketch, we first show that  $W(s) - \sigma^2(s)/2$  under  $\mathbb{Q}$  has the same distribution as  $W(s) - \gamma(s - t) + \sigma^2(t)/2$  under  $\mathbb{P}$ . The full proof requires doing this calculation for finite-dimensional distributions to conclude that  $(W(s) - \sigma^2(s)/2)_{s \in \mathbb{R}^d}$  under  $\mathbb{Q}$  has the same distribution as  $(W(s) - \gamma(s - t) + \sigma^2(t)/2)_{s \in \mathbb{R}^d}$  under  $\mathbb{P}$ , but doesn't require additional insights. We compare generating functions: for any  $\beta \in \mathbb{R}$ ,

$$\begin{aligned} & \log \mathbb{E}^\mathbb{Q} \exp(\beta(W(s) - \sigma^2(s)/2)) \\ &= -\frac{1}{2}\sigma^2(t) - \frac{\beta}{2}\sigma^2(s) + \frac{1}{2}\mathbb{V}\text{ar}[W(t) + \beta W(s)] \\ &= -\frac{\beta}{2}\sigma^2(s) + \beta\text{Cov}(W(t), W(s)) + \frac{1}{2}\mathbb{V}\text{ar}[\beta W(s)] \\ &= \beta \left[ \frac{1}{2}\sigma^2(t) - \gamma(s - t) \right] + \frac{1}{2}\mathbb{V}\text{ar}[\beta W(s)] \\ &= \beta \mathbb{E} \left[ W(s) - \gamma(s - t) + \frac{1}{2}\sigma^2(t) \right] + \frac{\beta^2}{2} \mathbb{V}\text{ar} \left[ W(s) - \gamma(s - t) + \frac{1}{2}\sigma^2(t) \right]. \end{aligned}$$

Since  $F$  is translation invariant, the  $F$ -value of  $(W(s) - \gamma(s - t) + \sigma^2(t)/2)_{s \in \mathbb{R}^d}$  must be the same as the  $F$ -value of  $(W(s) - W(t) - \gamma(s - t))_{s \in \mathbb{R}^d}$ . The latter has the same distribution as  $(Z(s - t))_{s \in \mathbb{R}^d}$ , which yields the claim.  $\square$

*Proof of Theorem 2.1.* Let  $t_i \in \mathbb{R}^d, i = 1, \dots, n$  and  $y_i \in \mathbb{R}, i = 1, \dots, n$  be arbitrary. From Lemma 5.1 with  $X_i = W_i - \sigma^2/2$  we deduce that

$$\mathbb{P}(\eta(t_1) \leq y_1, \dots, \eta(t_n) \leq y_n) = \exp \left( - \mathbb{E} \exp \left( \max_{j=1, \dots, n} (W(t_j) - \sigma^2(t_j)/2 - y_j) \right) \right).$$

Suppose that  $\mu$  is an arbitrary probability measure on  $\mathbb{R}^d$ . Applying Lemma 5.2 with

$$F(x) = \frac{\max_{j=1, \dots, n} \exp(x(t_j) - y_j)}{\int_{\mathbb{R}^d} \exp(x(s)) \mu(ds)},$$

we find that

$$\begin{aligned}
& \mathbb{E} \exp \left( \max_{j=1, \dots, n} (W(t_j) - \sigma^2(t_j)/2 - y_j) \right) \\
&= \int_{\mathbb{R}^d} \mathbb{E} \left[ \exp(W(t) - \sigma^2(t)/2) \frac{\exp \left( \max_{j=1, \dots, n} (W(t_j) - \sigma^2(t_j)/2 - y_j) \right)}{\int_{\mathbb{R}^d} \exp(W(s) - \sigma^2(s)/2) \mu(ds)} \right] \mu(dt) \\
&= \int_{\mathbb{R}^d} \mathbb{E} \left[ \frac{\exp \left( \max_{j=1, \dots, n} (Z(t_j - t) - y_j) \right)}{\int_{\mathbb{R}^d} \exp(Z(s - t)) \mu(ds)} \right] \mu(dt) \\
&= \mathbb{E} \left[ \frac{\exp \left( \max_{j=1, \dots, n} (Z(t_j - T) - y_j) \right)}{\int_{\mathbb{R}^d} \exp(Z(s - T)) \mu(ds)} \right],
\end{aligned}$$

where the last expectation is taken with respect to the distribution of  $(T, Z)$ , which is the product of the marginals. Applying Lemma 5.1 with

$$X_i(t) = Z_i(t - T_i) - \log \left( \int_{\mathbb{R}^d} \exp(Z_i(s - T_i)) \mu(ds) \right)$$

shows that

$$\mathbb{P}(\eta(t_1) \leq y_1, \dots, \eta(t_n) \leq y_n) = \mathbb{P}(\zeta(t_1) \leq y_1, \dots, \zeta(t_n) \leq y_n).$$

This yields the claim of Theorem 2.1. □

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#### APPENDIX A. COMPUTER CODE

This Matlab code is for 1-dimensional parameter spaces, but it is almost immediately adaptable for use with random fields due to Matlab's capabilities to work with multidimensional arrays. We present the Matlab code here since it can be read as pseudo-code, while reading the R code requires some knowledge of R objects designed for spatial data.

```

function res = generate_cluster(n,V)
    T = floor(n*rand());
    W = generateWwithdriftandcenter(T);
    res = V + W - log(sum(exp(W)));
end

function supremum = maxstable(n)
    supremum = -Inf(n,1);
    expminusV = -log(rand())/n;
    C = generate_cluster(n,-log(expminusV));

    while ( min(max(supremum, C)) < -log(expminusV) )
        supremum = max(supremum, C);
        expminusV = expminusV - log(rand())/n;
        C = generate_cluster(n,-log(expminusV));
    end

```

```

supremum = max(supremum, C);
end

```

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